ELECTRIC CURRENT AND MAGNETOSTATICS

In the previous chapter our centre of study was charge at rest. This chapter covers the study of charges in motion. The motion of charges is visualised as electrons flowing through a conductor. The rate at which charge flows through the given cross section of a conductor is called **electric current**.

If current does not change with time we call it **steady current**. If q is the amount of charge flowing through any cross section of a given conductor in time t, then current I is given by

$$I = \frac{q}{t}$$

If the flow of charge changes with time we can give instantaneous current by

$$I = \frac{dq}{dt}$$

The conventional flow of current is taken along the direction of flow of positive charge. Positive charges in a conductor are immobile, what flows physically are the electrons. So the direction of conventional current is opposite to the flow of electrons in the conductor. Electric current is a scalar quantity because it adds up like scalars.

Current Density

The current density at any point is defined as the amount of charge per unit time that flows through a unit area of the given cross section of a conductor. In other words it is defined as the amount of current flowing per unit area of cross section of the conductor. Current density is a vector quantity whose direction is same as the motion of positive charges.

If dI is the current flowing normal to cross sectional area da of the conductor, then current density is given by

$$J = \frac{dI}{da}$$

$$dI = J\,da \Rightarrow I = \iint J \cdot da \tag{1}$$

If current density J and area vector makes angle θ between them then the above equation takes that component of da which is along J, thus we have

$$dI = J \cdot da = J da \cos\theta \tag{2}$$

Equation of continuity-conservation of charge

The conservation of charge can be expressed in the form of a mathematical equation. That equation is called as equation of continuity.

Rewriting equation (2), we have

$$dI = J \cdot da$$

From here the total current flowing through the closed surface is given by

$$I = \oiint J \cdot da \tag{3}$$

If the volume charge density is ρ, the total charge inside the given volume is

$$q = \iiint \rho \, d\tau \tag{4}$$

The conservation of charge states that charge is never lost or created, so if there is a net current out of a closed surface the amount of charge inside the volume must decrease by the same rate. We can therefore write

$$\oiint J \cdot da = -\frac{dq}{dt} \tag{5}$$

Therefore we have

$$\oiint J \cdot da = -\frac{d}{dt} \iiint \rho \, d\tau$$

$$\oiint J \cdot da = -\iiint \frac{\partial \rho}{\partial t} d\tau \tag{6}$$

The Gauss divergence theorem to the L.H.S of the above equation makes it

$$\oiint J \cdot da = \iiint (\nabla \cdot J) d\tau$$

Then equation (6) can be written as

$$\iiint (\nabla \cdot J) d\tau = -\iiint \frac{\partial \rho}{\partial t} d\tau$$

Since this is true for any volume, the integrants must be equal

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t} \tag{7}$$

This is called the equation of continuity. It tells us about the **local conservation of charge**. The total charge in the universe is constant is the global conservation of charge. Local conservation of charge means that if total charge within some volume changes, exactly that amount of charge must have passed into or out of the volume.

Ohm's law and electrical conductivity

For current to flow through a conductor a potential difference must be applied across its ends. If potential difference V is applied across the ends of a conductor causes current I to flow through it then conventional form of Ohm's law is written as

$$V = IR \tag{8}$$

Where R is called the resistance of a conductor which is defined as the opposition offered by the conductor to the flow of current through it.

Resistance of a conductor of unit length and unit area of cross section is called **electric resistivity (**simply **resistivity)**. Resistivity depends on the nature of the substance and it characterises materials on the basis of how well current flows through them. Mathematically it is represented by ρ and expressed by the formula

$$R = \rho \frac{l}{A} \tag{9}$$

Here l is the length of the conductor and A is its cross sectional area. Putting $l = 1$ and $A = 1$ you get the definition of resistivity.

The reciprocal of resistivity is called **electric conductivity** (denoted by σ). Therefore $\sigma = \frac{1}{\rho}$. Both electric conductivity and electric resistivity gives the ease with which current flows through a conductor. A material having low electric conductivity (insulators) will have high value of electric resistivity.

Microscopic form of Ohm's law

Consider a conductor having length l and area of cross section A. If we applying a potential difference V across its ends suppose current I flows through it. Then the current density is given by

$$J = \frac{I}{A} \tag{10}$$

The electric field intensity is given by

$$E = \frac{V}{l} \tag{11}$$

Using equation (9) in equation (8), we get

$$V = I\rho \frac{l}{A}$$

$$\frac{V}{l} = \rho \frac{I}{A}$$

This equation on using equation (10) and (11) gives

$$E = \rho J$$

$$J = \frac{1}{\rho} E$$

But $\frac{1}{\rho} = \sigma$, therefore we have

$$J = \sigma E \tag{12}$$

This is the microscopic form of Ohm's law. This form of Ohm's law operates on a microscopic level relating current density, electric conductivity and electric field intensity.

Limitations of Ohm's law

1. This law is not applicable to semiconductors like silicon and germanium (also called non-ohmic conductors).
2. This law is not applicable to diodes and transistors. The voltage current graphs for diodes and transistors are not linear as expected from Ohm's law.
3. This law is not applicable to non-linear elements. Non-linear elements are those in which the resistance value changes with applied voltage. For example thyristor, electric arc etc.
4. If the physical conditions and temperature changes Ohm's law is violated.

Introduction to magnetostatics-magnetic fields

An experiment was done in which two wires carrying current in opposite direction were held near one another. It was found that the wires repel each other. However if the current in both wires flow in the same direction the wires attract each other. The first explanation that comes to mind for this attraction and repulsion is that the agency driving the current (battery) is charging up the wire and the force is due to electric charges. But if a test charge is held near these wires there is no force on the test charge. However if a magnetic needle is placed near these wires it shows deflection. This means the current carrying wires have produced a magnetic field to cause deflection in the magnetic needle. The magnetic field produced by one wire encounters the magnetic field produced by the other and hence the force of attraction or repulsion is observed. Since current is caused by some charge in motion we conclude that *a moving charge, in addition to electric field, produces Magnetic Field.*

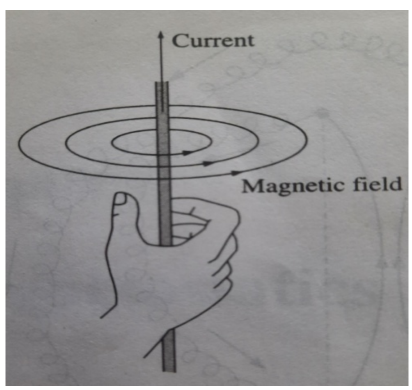


Figure 1

The magnetic field produced by current carrying wire circles around the wire. If you hold the wire in your right hand with the thumb pointing in the direction of current the curl of your fingers gives the direction of magnetic field (see figure 1 above). Such a field leads to force on a nearby current carrying wire. We will discuss about that force below.

Magnetic force

The magnetic force on a charge q moving with velocity v in presence of magnetic field B is

$$F_m = q(v \times B) \tag{13}$$

The direction of F_m would be the direction of $v \times B$. The rule finding this direction is: *stretch the middle finger, index finger and the thumb of your right hand in such a way that they are mutually perpendicular to each other. If the middle finger points in the direction of magnetic field, the index finger points in the direction of velocity of charge then the thumb points in the direction of magnetic force.* Applying this rule for the wire carrying current, the direction of force is as shown in figure 2. One can then explain the repulsion and attraction in wires carrying current in the same direction and opposite direction.

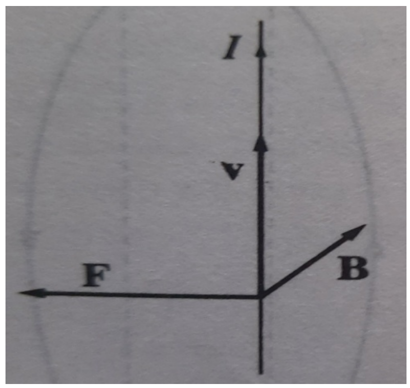


Figure 2

Equation (13) can be rewritten as

$$F_m = qvB \sin\theta \tag{14}$$

Where θ is the angle between v and B. Note that if $v = 0$ and $\theta = 0°$ or $180°$ $F_m = 0$.

So a particle at rest placed in magnetic field experiences no magnetic force. Also particle moving parallel or anti-parallel to the magnetic field experiences no magnetic force. Maximum magnetic force is experienced by the charged particle if it moves at right angles to the direction of magnetic field. And the maximum force is $F_m = qvB$. This is because $\sin 90° = 1$ (max).

If the charge q moves a distance dl in time dt, then $dl = v\,dt$. Work done by magnetic force is

$$dW_m = F_m \cdot dl = q(v \times B) \cdot vdt = 0 \tag{15}$$

This is because vector law makes $(v \times B) \cdot v = 0$. Thus we have

Magnetic forces do no work.

A magnetic field can change the direction of motion of the charged particle but it cannot accelerate the particle. If a charged particle is placed in magnetic field and electric field at the same time it experiences a force called **Lorentz force** given by

$$F = F_e + F_m = qE + q(v \times B) = q(E + v \times B) \tag{16}$$

In presence of both fields the charged particle is accelerated by the electric field and its path is altered by the magnetic field. This effect is used in a cyclotron to accelerate particles to very high speeds.

Biot-Savart law

This law is used to find magnetic field at any point due to a **steady current** flowing in a conductor of some arbitrary shape. Steady current produces constant magnetic field analogous to stationary charges producing constant electric fields. Steady current is a flow of charge which goes forever without any change and without accumulation of charge anywhere.

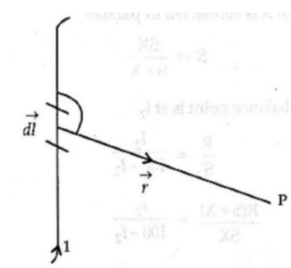


Figure 3

According to Biot-Savart law, the magnetic field dB at any point P due to a small current element of length dl of the conductor XY carrying current I is given by

$$dB = \frac{\mu_0}{4\pi}\frac{Idl \times \hat{r}}{r^2} \tag{17}$$

Where $\vec{r} = r\hat{r}$ is the position vector of point P in the direction of $\hat{r}$.

If the angle between dl and $\hat{r}$ is θ, we have

$$dB = \frac{\mu_0}{4\pi}\frac{Idl\sin\theta}{r^2}$$

The magnetic field due to the whole length of the conductor is then given by

$$B = \int dB = \frac{\mu_0}{4\pi}\int \frac{Idl \times \hat{r}}{r^2} \tag{18}$$

The integration is along the whole path in the direction of current. The constant μ_0 is called **permeability of free space** and is an exact number having value $\mu_0 = 4\pi \times 10^{-7} N/A^2$.

Magnetic field is measured in the units of Newton per ampere meter or **tesla** (**T**).

$$1T = 1N/Am$$

Biot-Savart law plays the same role in magnetostatics as Coulomb's law play in electrostatics. Both laws have the $\frac{1}{r^2}$ dependence.

Magnetic field due to an infinite straight wire carrying current-An application of Biot-savart law

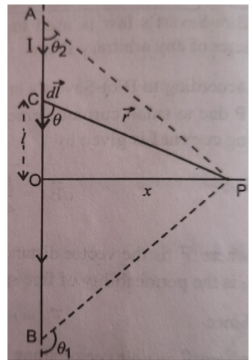


Figure 4

Consider a section AB of an infinite wire carrying current I from A to B as shown in figure 4. We wish to find magnetic field at point P at a distance of r from a current element dl of the wire. Let $OP = x$ be the perpendicular distance from P on the wire. According to the Biot-Savart law magnetic field at point P will be

$$dB = \frac{\mu_0}{4\pi}\frac{Idl\sin\theta}{r^2} \tag{19}$$

From the triangle POC we have $r = x\operatorname{cosec}\theta$ and $l = x\cot\theta \Rightarrow dl = -x\operatorname{cosec}^2\theta\, d\theta$

Therefore equation (19) becomes

$$dB = \frac{\mu_0}{4\pi}\frac{I(-x\operatorname{cosec}^2\theta\, d\theta)\sin\theta}{x^2\operatorname{cosec}^2\theta} = \frac{\mu_0 I}{4\pi x}(-\sin\theta)d\theta$$

Total magnetic field at point P due to whole length of the conductor is

$$B = \int dB = \frac{\mu_0 I}{4\pi x}\int_{\theta_1}^{\theta_2}(-\sin\theta)d\theta = \frac{\mu_0 I}{4\pi x}(\cos\theta_2 - \cos\theta_1) \tag{20}$$

If the conductor is infinitely long $\theta_1 = \pi$ and $\theta_2 = 0$. Therefore $\cos\theta_1 = -1$ and $\cos\theta_2 = 1$

Equation (20) then becomes

$$B = \frac{\mu_0 I}{2\pi x} \tag{21}$$

Just like stationary charge produces electric field which does not vary with time steady current produces magnetic field which does not vary with time. We will study time varying fields in the next chapter. Here we are concerned with steady currents only and an infinitely long wire carrying current is the case of conductor having steady currents because for an infinitely long wire the current will flow forever without the accumulation of charge anywhere. Practically we do not have such long infinite conductors but a current loop does the job.

Ampere's Circuital Law-Curl of B

This law (also simply called Ampere's law) states that the line integral of magnetic field around any closed path is equal to μ_0 times the current enclosed by the closed path. The closed path chosen is called Amperian loop.

Mathematically

$$\oint B \cdot dl = \mu_0 I$$

Proof: Consider a circular Amperian loop of radius r around an infinitely long wire carrying current I as shown in figure 5. The magnetic field at any point on the Amperian loop will be given by equation (21). That is

$$B = \frac{\mu_0 I}{2\pi r} \tag{22}$$

We know that the magnetic field of a straight wire carrying current curls around the wire, the direction of B at any point on the loop will be same as that of a small element dl of the loop. So the angle between B and dl will be zero. Therefore $B \cdot dl = Bdl \cos 0 = Bdl$.

Now the line integral of B around the circular loop will be

$$\oint B \cdot dl = \oint Bdl = \oint \frac{\mu_0 I}{2\pi r} dl = \frac{\mu_0 I}{2\pi r} \oint dl = \frac{\mu_0 I}{2\pi r} \times 2\pi r = \mu_0 I$$

Or

$$\oint B \cdot dl = \mu_0 I \tag{23}$$

Equation (23) is the **integral form** of Ampere's law. This law plays the same role in magnetostatics as Gauss's law does in electrostatics.

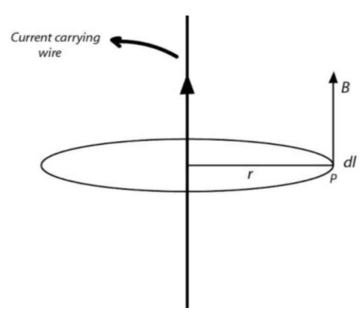


Figure 5

Go to equation (1) which is the relation between current and current density, rewriting it we have

$$I = \iint J \cdot da \tag{24}$$

From Stoke's theorem, we have

$$\oint B \cdot dl = \iint (\nabla \times B) \cdot da \tag{25}$$

With equations (24) and (25) equation (23) becomes

$$\iint (\nabla \times B) \cdot da = \mu_0 \iint J \cdot da = \iint \mu_0 J \cdot da$$

Since this equation is true for all surfaces we put the integrants equal. Therefore

$$\nabla \times B = \mu_0 J \tag{26}$$

This equation gives the curl of B and is the **differential form** of Ampere's law. This equation cannot determine B uniquely because many values of B can satisfy this equation. To find B completely we require divergence of B.

Divergence of *B*

We know that magnetic field strength at any point due to a current carrying wire at a distance r from the wire is given by Biot-Savart law as [equation (18)].

$$B = \frac{\mu_0}{4\pi} \int \frac{Idl \times \hat{r}}{r^2}$$

Taking divergence of the above equation we have

$$\nabla \cdot B = \nabla \cdot \frac{\mu_0}{4\pi} \int \frac{Idl \times \hat{r}}{r^2} = \frac{\mu_0 I}{4\pi} \nabla \cdot \int dl \times \frac{\hat{r}}{r^2}$$

Since integration and divergence are independent of each other we can interchange the operators. So we have

$$\nabla \cdot B = \frac{\mu_0 I}{4\pi} \int \nabla \cdot \left(dl \times \frac{\hat{r}}{r^2} \right) \tag{27}$$

Using the vector indentity $\nabla \cdot (A \times B) = \mathrm{B} \cdot (\nabla \times A) - \mathrm{A} \cdot (\nabla \times B)$

Therefore we have

$$\nabla \cdot B = \frac{\mu_0 I}{4\pi} \int \left[\frac{\hat{r}}{r^2} \cdot (\nabla \times dl) - dl \cdot \left(\nabla \times \frac{\hat{r}}{r^2} \right) \right] \tag{28}$$

Since dl is not a function of co-ordinates (x, y, z) of point P, therefore $\nabla \times dl = 0$. We also have an expression from vector identities which is $\hat{r}/_{r^2} = -\nabla(1/_r)$. Therefore equation (28) becomes

$$\nabla \cdot B = \frac{\mu_0 I}{4\pi} \int dl \cdot \left(\nabla \times \nabla \left(\frac{1}{r} \right) \right)$$

But $\nabla \times \nabla\left(\frac{1}{r}\right) = 0$ as the curl of gradient is always zero. Therefore we have

$$\nabla \cdot B = 0 \tag{29}$$

Thus the divergence of B is always zero.

We know that electric field lines begin from a positive charge and end on a negative charge. Mathematically this is given by divergence of E (Gauss law), which is non-zero. But magnetic field lines do not originate or terminate anywhere. They either form closed loops or extends up to infinity. For magnetic field lines to begin or end somewhere divergence of B must be non-zero which is not the case as proved above. *In other words there are no point sources for B as there are for E. That means magnetic mono-poles do not exist.* This is the physical meaning of divergence of B.

Magnetic Scalar and Vector Potentials

In electrostatics $\nabla \times E = 0$ enables us to write $E = -\nabla V$, where V is a scalar called electric potential. But in magnetostatics $\nabla \times B = \mu_0 J$, so it is not possible to write magnetic equivalent of electric potential. In special case when $J = 0$ (however this is incompatible with Ampere's law), we have $\nabla \times B = 0$ and we can write B as the negative gradient of a scalar quantity.

$$B = -\nabla U$$

Here U is called the **magnetic scalar potential**. We can make use of magnetic scalar potential to find B in cases where $J = 0$. But equation (29) introduces a vector potential in magnetostatics. This is because from vector analysis, we know that divergence of curl is zero. That is $\nabla \cdot (\nabla \times A) = 0$. Compared with equation (29) we have

$$B = \nabla \times A \tag{30}$$

Here A is called the **magnetic vector potential**. We can derive an expression for magnetic vector potential.

Expression for magnetic vector potential

We will start with the Biot-Savart law which is

$$dB = \frac{\mu_0}{4\pi}\frac{Idl \times \hat{r}}{r^2}$$

If we have a current loop the total magnetic field due to the whole loop is

$$B = \frac{\mu_0 I}{4\pi}\oint dl \times \frac{\hat{r}}{r^2} = -\frac{\mu_0 I}{4\pi}\oint dl \times \nabla\left(\frac{1}{r}\right)$$

Here we have made use of the expression $\hat{r}/_{r^2} = -\nabla(1/r)$. Now making use of the identity $A \times B = -B \times A$, we can write

$$B=\frac{\mu_0 I}{4\pi}\oint \nabla\left(\frac{1}{r}\right)\times dl$$

Using another vector identity $\nabla\varphi\times A=\nabla\times(\varphi A)-\varphi(\nabla\times A)$, we have

$$B=\frac{\mu_0 I}{4\pi}\oint\left\{\nabla\times\frac{dl}{r}-\frac{1}{r}(\nabla\times dl)\right\}$$

Since dl is not a function of (x,y,z), therefore $\nabla\times dl=0$. Thus we have

$$B=\frac{\mu_0 I}{4\pi}\oint \nabla\times\frac{dl}{r}$$

$$B=\nabla\times\oint\frac{\mu_0 Idl}{4\pi r}$$

Compare this with equation (30) we can write

$$A=\oint\frac{\mu_0 Idl}{4\pi r}=\frac{\mu_0}{4\pi}\oint\frac{Idl}{r} \tag{31}$$

This is the expression for magnetic vector potential. This can also be expressed in terms of current density J. We can write

$$Idl=\frac{I}{A}Adl=Jd\tau$$

Here $\frac{I}{A}=J$ is the current density, and $Adl=d\tau$ is the volume element. Therefore equation (31) becomes

$$A=\frac{\mu_0}{4\pi}\iiint\frac{J}{r}d\tau \tag{32}$$

Divergence of A

Taking the divergence of equation (32) we have

$$\nabla\cdot A=\nabla\cdot\frac{\mu_0}{4\pi}\iiint\frac{J}{r}d\tau=\frac{\mu_0}{4\pi}\iiint\nabla\cdot\frac{J}{r}d\tau$$

Using Gauss divergence theorem we have

$$\iiint\nabla\cdot\frac{J}{r}d\tau=\oiint\frac{J}{r}\cdot da$$

Thus we have

$$\nabla\cdot A=\frac{\mu_0}{4\pi}\oiint\frac{J}{r}\cdot da \tag{33}$$

The integration of the above equation can be extended and considered over the whole space extending upto infinity. For finite current distribution the current density J vanishes at infinity. We therefore have

$$\oiint \frac{J}{r} \cdot da = 0$$

This gives from equation (33)

$$\nabla \cdot A = 0 \tag{34}$$

Thus the divergence of magnetic vector potential is zero. This is a useful result and we will use it to write *Ampere's law in terms of magnetic vector potential*.

The Ampere's law is

$$\nabla \times B = \mu_0 J$$

Using equation $B = \nabla \times A$ we can write

$$\nabla \times (\nabla \times A) = \mu_0 J$$

$$\nabla(\nabla \cdot A) - \nabla^2 A = \mu_0 J$$

Left hand side of this equation used the vector identity $\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$. Now using equation (34) the above equation takes the form

$$\nabla^2 A = -\mu_0 J \tag{35}$$

This expression is the Ampere's law in terms of A. We also call this as Poisson equation for vector potential.

Derivation of Biot-Savart Law from Magnetic Vector Potential

The magnetic vector potential is defined by the expression

$$A = \frac{\mu_0}{4\pi} \oint \frac{Idl}{r}$$

Now since $B = \nabla \times A$, so we have

$$B = \nabla \times \frac{\mu_0}{4\pi} \oint \frac{Idl}{r} = \frac{\mu_0 I}{4\pi} \oint \nabla \times \frac{dl}{r}$$

Now we will make use of the identity $\nabla \times (\varphi A) = \varphi(\nabla \times A) + \nabla\varphi \times A$, and write the above equation as (with $\varphi = {}^1\!/_r$ and $A = dl$).

$$B = \frac{\mu_0 I}{4\pi} \oint \nabla \times \frac{dl}{r} = \frac{\mu_0 I}{4\pi} \oint \left\{\frac{1}{r}(\nabla \times dl) + \nabla\left(\frac{1}{r}\right) \times dl\right\}$$

Since dl is not a function of (x, y, z), we have $\nabla \times dl = 0$. Also ${}^{\hat{r}}\!/_{r^2} = -\nabla({}^1\!/_r)$. Therefore

$$B = -\frac{\mu_0 I}{4\pi} \oint \frac{\hat{r}}{r^2} \times dl = \frac{\mu_0 I}{4\pi} \oint dl \times \frac{\hat{r}}{r^2}$$

This is Biot-Savart law for a closed loop. On differentiating the above equation we get Biot-Savart law in the form

$$dB = \frac{\mu_0}{4\pi}\frac{Idl \times \hat{r}}{r^2}$$

Modified form of Ampere's Law

Ampere's law fails beyond magnetostatics and this requires a modification to the law. Maxwell fixed Ampere's law and the modified form of the law is thus called **Maxwell-Ampere law**. First we look at the flaw in the law and then see how it was fixed. As we know the Ampere's law has the form

$$\nabla \times B = \mu_0 J$$

Applying the divergence to this equation we write

$$\nabla \cdot (\nabla \times B) = \mu_0 \nabla \cdot J \tag{36}$$

Since the divergence of curl is zero the L.H.S of the above equation is zero. Let's look at the R.H.S of the above equation. Recall the equation of continuity $\nabla \cdot J = -\frac{\partial \rho}{\partial t}$, for steady currents ρ is a constant and thus divergence of J is zero and equation (36) is satisfied, but when we go beyond steady currents (where ρ varies with time) the divergence of J is not zero. Therefore equation (36) is valid only for steady currents, that is, Ampere's law cannot be right beyond magnetostatics. Maxwell suggested that something must be added to J in the Ampere's law. With this suggestion Ampere's law can be written as

$$\nabla \times B = \mu_0 (J + J_d) \tag{37}$$

To find what J_d is let's apply divergence to eq. (37)

$$\nabla \cdot (\nabla \times B) = \mu_0 \nabla \cdot (J + J_d) = \mu_0 (\nabla \cdot J + \nabla \cdot J_d)$$

But $\nabla \cdot (\nabla \times B) = 0$, so we have

$$\mu_0 (\nabla \cdot J + \nabla \cdot J_d) = 0$$

$$\nabla \cdot J + \nabla \cdot J_d = 0$$

$$\nabla \cdot J = -\nabla \cdot J_d \tag{38}$$

From equation of continuity we have

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t}$$

Therefore

$$\nabla \cdot J_d = \frac{\partial \rho}{\partial t} \tag{39}$$

From Gauss law in electrostatics, we have

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \Rightarrow \rho = \epsilon_0 \nabla \cdot E$$

Therefore equation (39) becomes

$$\nabla \cdot J_d = \frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot E) = \nabla \cdot \epsilon_0 \frac{\partial E}{\partial t}$$

Thus

$$J_d = \epsilon_0 \frac{\partial E}{\partial t} \tag{40}$$

Putting this value in equation (37), we get the Maxwell-Ampere's law or the modified form of Ampere's law.

$$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \tag{41}$$

Maxwell called this extra term the **displacement current density.** As seen from equation (40) the cause of this term is the time varying electric field. Looking at equation (41) this time varying electric field gives rise to magnetic field. Maxwell fixed Ampere's law purely on theoretical basis with the resulting argument.

A changing electric field induces a magnetic field.

This introduces the concept of time varying fields. We will discuss more about time varying fields in the next chapter. Note that when the electric field is not changing with time we still have $\nabla \times B = \mu_0 J$, because in that case $\frac{\partial E}{\partial t} = 0$. Thus the modification changes nothing as far as magnetostatics is concerned.

In analogy with the conventional current density (current per unit area) we can define displacement current density as

$$J_d = \frac{I_d}{A} \Rightarrow I_d = J_d A = \epsilon_0 A \frac{\partial E}{\partial t}$$

I_d is called **displacement current**. It differs from the conventional current and the cause of this current is the time varying electric field. Like conventional current this current is also capable of producing magnetic field.

In integral form Maxwell-Ampere's law can be expressed as

$$\oint B \cdot dl = \mu_0 (I + I_d) \tag{42}$$

Current Loop as a Magnetic Dipole

A magnetic dipole is a pair of magnetic poles of opposite strength separated by a small distance. When current passes through a circular coil it acts as a magnetic dipole. When we look at one face of the loop we see the current flowing anticlockwise (say). This face of the loop acts as the north pole of the loop. The other opposite face of the loop where current is flowing in the clockwise direction acts as the south pole of the loop. Hence a current loop acts as a magnetic dipole with its magnetic dipole moment(P_m) directly proportional to

1. Current I flowing through the loop.
2. Area a of the loop

Therefore $$P_m \propto Ia$$

$$P_m = Ia \tag{43}$$

The constant of proportionality is taken as 1. If the loop has N turns we write

$$P_m = NIa \tag{44}$$

Relation Between Orbital Angular Momentum and Magnetic Dipole Moment-Atomic Orbits

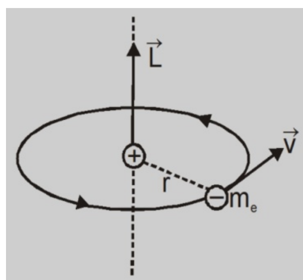


Figure 6

An atom consists of electrons revolving around the nucleus in fixed orbits. When an electron revolves in its orbit it constitutes a current (see figure 6). Note that the arrow on the loop is the direction of motion of electron, the direction of current will be opposite to it. Thus the atomic orbit is a current carrying loop and behaves as a magnet. Let's assume the orbit to be a circle or radius r and the electron moves with a velocity of v. The time period of the motion of electron is

$$T = \frac{2\pi r}{v}$$

So the magnitude of current in the orbit will be (current is charge per unit time)

$$I = \frac{e}{T} = \frac{ev}{2\pi r} \tag{45}$$

Here e is the charge of the electron. Since the area of the loop is $a = \pi r^2$, the magnetic dipole moment of the loop will be

$$P_m = Ia$$

$$P_m = \frac{ev}{2\pi r} \times \pi r^2 = \frac{evr}{2} = \frac{e}{2m} L$$

Here $L = mvr$ (m being the mass of electron) is the angular momentum of the electron. Therefore we have

$$P_m = \frac{e}{2m} L \tag{46}$$

This is the required relation. The ratio of magnetic dipole moment to the angular momentum is called **gyromagnetic ratio** (g). That is

$$g = \frac{P_m}{L} = \frac{e}{2m} = \frac{1.6 \times 10^{-1}}{2 \times 9.1 \times 10^{-31}} = 9 \times 10^{11} C/kg$$

The gyromagnetic ratio is a constant and is independent of shape and size of the orbit.

Here we can also define the unit for measurement of atomic magnetic moments called as **Bohr magneton.** According to the Bohr's model of atom, only those atomic orbits are possible for which the angular momentum is given by

$$L = \frac{nh}{2\pi}$$

Where $n = 1,2,3$ ... and h is the Planck's constant. Therefore equation (46) can be written as

$$P_m = \frac{e}{2m} \cdot \frac{nh}{2\pi} = n \cdot \frac{eh}{4\pi m}$$

If $n = 1$, we have

$$P_m = \frac{eh}{4\pi m} \tag{47}$$

This least value of magnetic moment is called Bohr magneton. Its value is 9.27×10^{-24} Am^2.

Magnetic Fields in Matter-Magnetisation

If we look at materials at the atomic scale we would find electron revolving in orbits around the nucleus. In addition to the orbital motion of electrons we also find electrons spinning about their axes. These motions constitute tiny current loops and we may treat them as magnetic dipoles. In absence of an external magnetic field these magnetic dipoles cancel each other's effect because of random orientation of atoms. Thus we do not have a net magnetic moment in the material. But when magnetic field is applied, the orientation no longer remains random but there is a net magnetic dipole moment in the material. Such a material is said to have been **magnetised** and the process is called **magnetisation.**

Suppose B is the external magnetic field. Materials can acquire magnetisation wherein the net magnetic moment is either parallel to B or opposite to B. In the first case the magnetised material is called a **paramagnet** and in the second case the magnetised material is called a **diamagnet**. Once the external magnetic field is removed these materials lose their paramagnetic and diamagnetic properties. There is another class of materials which retain the magnetic properties even after the removal of the external magnetic field called **ferromagnets**. Whatever may be the cause of magnetisation, we define the state of magnetisation by a vector M called the **magnetisation vector.** It is the measure of extent or degree to which some material is magnetised.

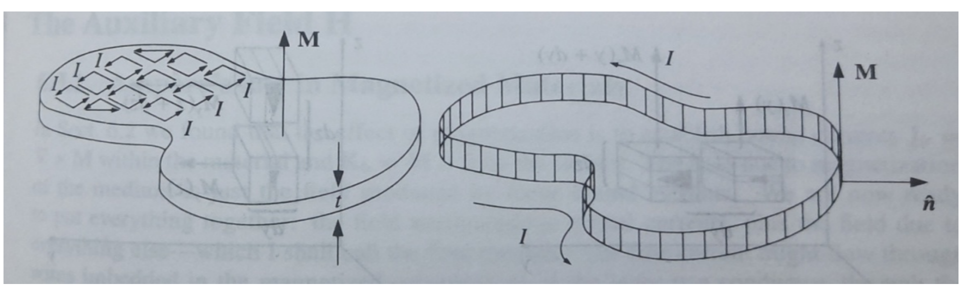
Magnetisation vector is defined as the net dipole moment per unit volume.

$$M = \frac{P_m}{V} \tag{48}$$

Magnetisation vector plays the role analogous to polarisation vector in electrostatics.

Bound currents

Consider a thin slab of a material magnetised uniformly having magnetisation M. The dipoles in the slab are represented by tiny current loops (figure 7a). When we look at the loops all the internal currents cancel. This is because at every point within the slab the currents flowing in the adjacent loops are oppositely directed and cancel each other's effect. However at the edges there are no adjacent loops to make the cancelling. The net result is that there is a net current flowing only along the boundary of the slab (figure 7b). Notice that no charge has made a single trip around the slab (however each charge has moved in the tiny loops) we still have a current along the boundary. Since the charge is attached to an atom we call this current as **surface bound current**. The corresponding current density is represented by K_b. Subscript b is used for the word bound. This current produces magnetic field in the same way as the ordinary current does.



(a) (b)

Figure 7

Suppose that each tiny loop has thickness t and surface area a (figure 8). Then its dipole moment will be (equation 48)

$$P_m = MV = Mat$$

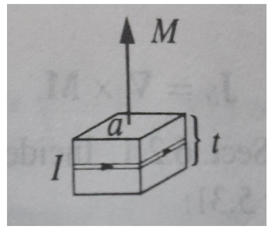


Figure 8

Also in terms of current carrying loops, referring to equation (43), we have

$$P_m = Ia$$

Thus we have $I = Mt$. So the **surface bound current density** will be current per unit length, that is

$$K_b = \frac{I}{t} = M$$

$$K_b = M \tag{49}$$

In vector form $K_b = M \times \hat{n}$. Here $\hat{n}$ is the outward drawn unit vector.

When the magnetisation is non-uniform the internal currents no longer cancel and we have finite values of these currents. We call these **volume bound currents**. The corresponding current density is represented by J_b. In this case two adjacent loops will be magnetised differently and would have different values of magnetisation vectors. Consider two adjacent loops with a larger arrow on one showing slightly greater magnetisation in the z-direction (figure 9a). On the surface where they join we have a net current in x-direction given by

$$I_x = net\ magnetisation \times thickness\ of\ the\ loop$$

$$I_x = [M_z(y + dy) - M_z(y)]dz = \frac{\partial M_z}{\partial y} dydz$$

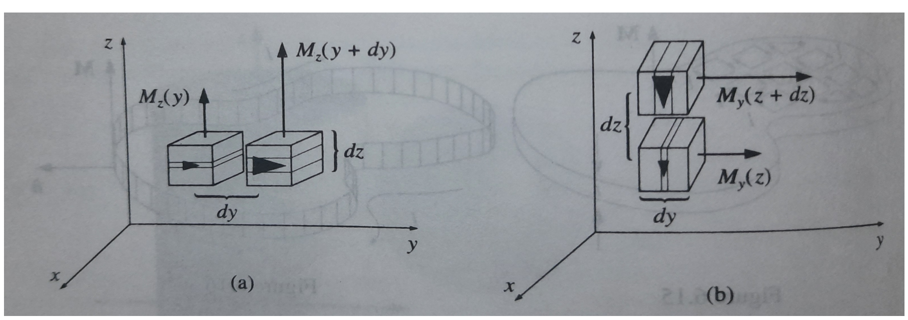


Figure 9

The corresponding volume current density (current per unit area) is therefore

$$(J_b)_x = \frac{I_x}{dydz} = \frac{\partial M_z}{\partial y}$$

Similarly the non-uniform magnetisation in the y-direction (figure 9b) will contribute an amount of $-\partial M_y/\partial z$. So the net current density in x-direction is

$$(J_b)_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}$$

Similarly current densities along y and z directions would be

$$(J_b)_y = \frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x}$$

$$(J_b)_z = \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y}$$

So the total current density is then

$$J_b = (J_b)_x\hat{\imath} + (J_b)_y\hat{\jmath} + (J_b)_z\hat{k}$$

$$J_b = \left(\frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y}\right)\hat{k}$$

$$J_b = \nabla \times M \tag{50}$$

As divergence of the curl is always zero we have

$$\nabla \cdot J_b = 0$$

Thus the effect of magnetisation is to introduce bound currents J_b within the material and K_b on the surface. The field due to magnetisation is just the field produced by these currents. Any current which is not a result of magnetisation is called **free current**. Currents flowing through wires within the magnetised substance or through the substance itself are free currents.

Ampere's Law in Magnetised Materials

We know that Ampere's law ha the form $\nabla \times B = \mu_0 J$. In a magnetised material we have two currents- one is the bound current which is a result of magnetisation and the other is free current because of the wires being connected to a battery. Therefore the total current can be written as sum of the free current and the bound current

$$J = J_f + J_b \tag{51}$$

Thus the Ampere's law becomes

$$\nabla \times B = \mu_0\left(J_f + J_b\right) = \mu_0\left(J_f + \nabla \times M\right)$$

Here we have made use of equation (50). Rearranging the above equation and putting together the two curls we have

$$\nabla \times \left(\frac{1}{\mu_0} B - M\right) = J_f$$

$$\nabla \times H = J_f \tag{52}$$

Where H is called auxiliary field and is the quantity in the parentheses given by

$$H = \frac{1}{\mu_0} B - M \tag{53}$$

Equation no (52) is the Ampere's law for magnetised materials. One can obtain its integral form also as follows.

Taking the surface integral of equation (52), we have

$$\iint (\nabla \times H) \cdot da = \iint J_f \cdot da$$

The right hand side of the above equation is the total current I_f passing through the Amperian loop. Applying Stokes theorem to the left hand side, we have the integral form of the Ampere's law as

$$\oint H \cdot dl = I_f \tag{54}$$

The introduction of H is very important. Firstly it plays the same role as D does in electrostatics. Secondly with H Ampere's law can be written in terms of free current alone. We can measure and control free current easily (just connect the wires with the battery and measure the current with the help of an ammeter) but cannot do the same with bound current. So while applying equation (54) we deal only with free current and this gives us the importance of auxiliary field H.

Magnetic Susceptibility and Magnetic Permeability

Magnetic susceptibility is defined as the ration of the magnetisation vector M to the auxiliary field H. It is denoted by χ_m.

$$\chi_m = \frac{M}{H} \tag{55}$$
$$M = \chi_m H$$

This equation results from the fact that M is proportional to H and χ_m stands as the proportionality constant. It is a dimensionless quantity and has different values for different magnetic materials. Its value is negative for diamagnets and positive for paramagnets. The values are typically very small but these values are very large and positive for ferromagnets. From equation (53) we have

$$B = \mu_0(H + M) = \mu_0(H + \chi_m H) = \mu_0(1 + \chi_m)H \tag{56}$$

Thus we see that B is also proportional to H. We can write this equation in the form

$$B = \mu H \tag{57}$$

Where

$$\mu = \mu_0(1 + \chi_m) \tag{58}$$

The factor μ is called the **magnetic permeability** of the material. For free space $\chi_m = 0$, because there is no material in free space to magnetise. Thus the permeability is $\mu = \mu_0$, which we already know is called **permeability of free space**.

Boundary conditions for B and H at the interface of two media

Consider a boundary separating two media having relative permeabilities μ_1 and μ_2. Consider a pill shaped Gaussian surface of height h across the boundary with surface area of top and bottom as da (figure 10). Suppose the media are placed in a magnetic field which has values B_1 and B_2 in the two media.

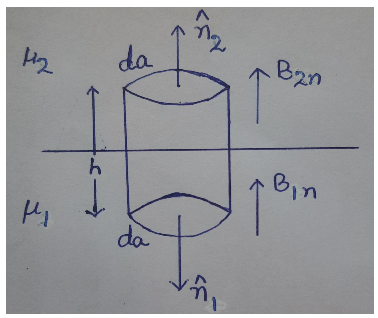


Figure 10

Normal components of B and H.

We know that

$$\nabla \cdot B = 0$$

Writing this in its integral form we will have

$$\oiint B \cdot da = 0$$

When we apply this equation for the Gaussian surface shown in the figure we divide it into three parts, top surface bottom surface and the curved surface. At the boundary the curved surface almost vanishes because in that case $h \to 0$ and hence its contribution to the above equation is zero. However for top and the bottom surface the equation can be written as

$$\left[\iint B_{1n} \cdot \hat{n}_1 da\right]_{top} + \left[\iint B_{2n} \cdot \hat{n}_2 da\right]_{bottom} = 0$$

Where $\hat{n}_1$ and $\hat{n}_2$ are the unit vectors in the directions as shown in the figure, from where it is clear that $\hat{n}_1 = -\hat{n}_2 = \hat{n}(say)$. B_{1n} and B_{2n} are the normal components of B. Therefore we have

$$\iint B_{1n} \cdot \hat{n} da - \iint B_{2n} \cdot \hat{n} da = 0$$

$$\iint (B_{1n} \cdot \hat{n} - B_{2n} \cdot \hat{n})\, da = 0$$

Since $da \neq 0$ and we have chosen an arbitrary surface, therefore we have

$$B_{1n} \cdot \hat{n} - B_{2n} \cdot \hat{n} = 0$$

$$B_{1n} \cdot \hat{n} = B_{2n} \cdot \hat{n}$$

$$B_{1n} = B_{2n} \tag{59}$$

Thus we see that the normal component of B is continuous across the boundary. Since we know that $B = \mu H$, we can write equation (59) as

$$\mu_1 H_{1n} = \mu_2 H_{2n} \tag{60}$$

Where H_{1n} and H_{2n} are the ***normal components*** of H. As $\mu_1 \neq \mu_2$ we have $H_{1n} \neq H_{2n}$ implying that the normal component of H is discontinuous across the boundary.

Tangential components of B and H

Ampere's law can be written in terms of H as follows. Ampere's law, as you know, for a medium having permeability μ has the form

$$\oint B \cdot dl = \mu I$$

Writing $B = \mu H$, the above equation takes the form

$$\oint H \cdot dl = I \tag{61}$$

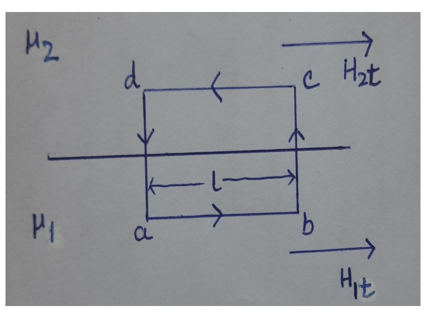


Figure 11

Now consider a rectangular loop $abcd$ across the boundary of the two media (figure 11). Taking the integral along the loop in the direction as shown in figure (Note that the choice of this direction is arbitrary, you can go around in the opposite direction and get the same result), we see that at the boundary the lengths bc and da approaches zero. Therefore the integrals along these paths will also be zero. That is $\int H \cdot dl|_{bc} = \int H \cdot dl|_{da} \to 0$. We therefore can write equation (61) as

$$\int H_{1t} \cdot dl\Big|_{ab} + \int H_{2t} \cdot dl\Big|_{cd} = I \tag{62}$$

Here H_{1t} and H_{2t} are the tangential components of H_1 and H_2 in medium 1 and 2 respectively. Equation (62) can be further written as

$$\int H_{1t}\, dl \cos 0\Big|_{ab} + \int H_{2t}\, dl \cos 180\Big|_{cd} = I$$

$$H_{1t} \int dl\Big|_{ab} - H_{2t} \int dl\Big|_{cd} = I$$

$$H_{1t} l - H_{2t} l = I$$

$$H_{1t} - H_{2t} = \frac{I}{l} = K_s \tag{63}$$

Note that lengths $ab = cd = l$. And current per unit length is called surface current density K_s. from equation (63) the tangential component of H is not continuous across the boundary. However in current free surfaces ($K_s = 0$) H is continuous across the boundary. In that case

$$H_{1t} = H_{2t}$$

Using equation $B = \mu H$ the above equation can be written in terms of B as

$$\frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2}$$

Here B_{1t} and B_{2t} are the tangential components of B_1 and B_2 in medium 1 and 2 respectively. As $\mu_1 \neq \mu_2$ we have $B_{1t} \neq B_{2t}$. Therefore the tangential component of B is not continuous across the interface.